Chapter 2

Optimal Control of ODEs: the Linear-Quadratic (LQ) case

In this chapter, the optimal control of systems governed by an ODE is studied quite briefly. We refer to [26] for a full course on the subject; the present chapter follows closely the presentation done in [26] 1.

First we present basic examples, next we define the notion of controllability and present the Kalman’s condition. The most simple but very important case is studied more in details: linear model - quadratic cost function (LQ case). In particular a proof of existence and uniqueness of the solution is given; and its characterization using the Pontryagin’s maximum principle and the Hamiltonian.

2.1 Introduction

In order to illustrate what is an optimal control problem let us give a simple example (extracted from a Wikipedia webpage).

Consider a car traveling on a straight line through a hilly road. The question is, how should the driver press the accelerator pedal in order to minimize the total traveling time? The control law refers specifically to the way in which the driver presses the accelerator and shifts the gears. The ”system” or ”direct model” is the car dynamics, and the optimality criterion is the minimization of the total traveling time. Control problems usually include additional constraints. For example the amount of available fuel might be limited, the speed limits must be respected, etc.

The ”cost function” is a mathematical functional giving the traveling time as a function of the speed, initial conditions and parameters of the system.

Another optimal control problem can be to find the way to drive the car so as to minimize its fuel consumption, given that it must complete a given course in a time not exceeding some amount. Yet another control problem can be to minimize the total monetary cost of completing the trip, given assumed monetary prices for time and fuel.

1 The students of INSA Toulouse GMM5 can refer too to the course of automatic by C. Bès, [8].
Let us notice that the optimal control problem may have multiple solutions (i.e. the solution may not be unique), if it exists...

**Direct modelling, next control...**

<table>
<thead>
<tr>
<th>Input Parameters – control variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>• e.g. material properties, empirical laws coefs</td>
</tr>
<tr>
<td>• Boundary conditions, Initial conditions</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Model (ODEs, PDEs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational Code</td>
</tr>
</tbody>
</table>

- Target - Observations, measurements - Stabilization

| Output: computed solution |

- Next: control of the system

**Figure 2.1:** Direct Modelling with input parameters. Towards the control of the system.

### 2.2 An example

#### 2.2.1 The controlled system

This classical example is treated in detail in [26], we refer to this book for more details; see also [8].

Let us consider a spring - mass system (see Figure 2.2). The mass \( m \) is submitted to a force \( f \), which supposed to be equal to: \(-[k_1(x - L) + k_2(x - L)^3]\). \( L \) is the spring length at rest, \( k_* \) are the spring stiffness parameters. We will apply an external force: \( u(t) \) (depending on time \( t \)). Given the external force, the spring position \( x(t) \) is described by the equation:

\[
m x''(t) + k_1(x(t) - L) + k_2(x(t) - L)^3 = u(t) \text{ for } t \geq 0
\]

It is the so-called *direct model*. It is an ordinary differential equation, linear if \( k_2 = 0 \), non-linear if not. The initial condition, \( x(0) = x_0, x'(0) = y_0 \), is given.
The problem we consider is as follows. Given an initial state \((x_0, y_0)\), find the "best" external force \(u(t)\) such that \(x(t) = L\) in a minimum time and under the constraint \(u(t) \leq 1\) (the external force is bounded).

\(u(t)\) is the control of the problem.

For sake of simplicity (and without any change of the nature of problem), we set: \(m = 1, k_1 = 1, L = 0\). Then in the phase space \((x, x')\), the direct model writes:

\[
X'(t) = AX(t) + f(X(t)) + Bu(t), \quad X(0) = X0.
\]

where \(X(t) = (x, x')(t)\) is the state of the system,

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = (0, 1)^T, \quad f = (0, -k_2 x^3)^T
\]

The model is a 1st order differential system, linear if \(k_2 = 0\), non-linear if not.

### 2.2.2 Different states of the system according to the control values

1) In the case \(u(t) = 0\) (no external action), the solution of the system satisfies:

\[
x''(t) + x(t) + 2x(t)^3 = 0
\]

It is the so-called Duffing’s equation. One can prove that its solution \(x(t)\) satisfies: \(x(t)^2 + x(t)^4 + x'(t)^2 = \text{Cste}\). All solutions are periodic and can be represented by an algebraic curve. We plot the phase diagram and the trajectory on Fig. 2.3.

2) In the case \(u(t) = -x'(t)\), hence we try to damp the spring, the model writes:

\[
x''(t) + x(t) + 2x(t)^3 + x'(t) = 0
\]

One compute numerically its solution and one plot the phase diagram and the trajectory on Fig. 2.4.

One could verify with Lyapunov's theory that the origin is asymptotically stable. One can notice that the spring position and velocity reach the equilibrium position in infinite time, and not in finite time, thus this control do not satisfies the original problem.
3) Let us set \( u(t) = -(x(t)^2 - 1)x'(t) \), then the model writes:

\[
x''(t) + x(t) + 2x(t)^3 + (x(t)^2 - 1)x'(t) = 0
\]

It is a Van Der Pol type equation.

One compute numerically two different solutions and we plot the corresponding phase diagram and the trajectories on Fig. 2.5.

One could prove with Lyapunov’s theory that there exists a periodic solution which is attractive (Figure 2.5), [26, 8].

These three examples of regimes illustrate the wide range of behaviors one can obtain on this a-priori very simple differential system, if changing the control term.
2.3 Controllability

2.3.1 Reachable sets

Let \( x_u(t) \) be the solution of the ODE system corresponding to a given control \( u(t) \) (\( x_u \) is supposed to exist and to be unique).

**Definition 2.3.1.** The set of reachable states from the initial state \( x_0 \) in time \( T > 0 \), is defined by:

\[
\text{Acc}(x_0, T) = \{ x_u(T), \text{ with } u \in L^\infty([0, T], \Omega) \}
\]

with \( \Omega \) a compact subset of \( \mathbb{R}^m \).

We set: \( \text{Acc}(x_0, 0) = x_0 \).

**Theorem 2.3.1.** Let us consider the following 1st order linear ODE system in \( \mathbb{R}^n \):

\[
x'(t) = A(t)x(t) + B(t)u(t) + r(t)
\]

Then \( \text{Acc}(x_0, t) \) is compact, convex, and varies continuously with \( t, t \in [0, T] \).

*Partial proof.* We prove only that \( \text{Acc}(x_0, t) \) is convex in the case \( u \in L^\infty([0, T], \Omega) \) with \( \Omega \) convex. We refer to [26] for the proof of the remaining results.

Let \( \Omega \) be convex. Let \( x_1^t, x_2^t \in \text{Acc}(x_0, t) \); we denote by \( u_i \) the corresponding controls. We have:

\[
x_i(0) = x_0 ; \quad x_i^t(t) = x_i(t) ; \quad x_i^t(s) = A(s)x_i(s) + B(s)u_i(s) + r(s) ; \quad i = 1, 2
\]

Let \( \lambda \in [0, 1] \). We seek to prove that: \( (\lambda x_1^t + (1 - \lambda)x_2^t) \in \text{Acc}(x_0, t) \).

We solve the 1st order linear ODE, we obtain:

\[
x_i^t = x_i(t) = M(t)x_0 + M(t) \int_0^t M(s)^{-1}B(s)u_i(s)ds
\]

with \( M(t) \in M_{n,n}(\mathbb{R}) \) such that: \( M'(t) = A(t)M(t), \quad M(0) = Id. \) (If \( A(t) = A \) constant then \( M(t) = \exp(tA) \).
We set: \( u(t) = \lambda u_1(t) + (1 - \lambda)u_2(t) \). Since \( \Omega \) is convex, \( u(t) \in \Omega \); also \( u \in L^\infty([0, T], \Omega) \).
Let \( x(t) \) be the solution associated to \( u(t) \). We have \( x(t) \in \text{Acc}(x_0, t) \) and:

\[
x(t) = M(t)x_0 + M(t)\int_0^t M(s)^{-1}B(s)u(s)ds
\]

In other respect,

\[
\lambda x_1^t + (1 - \lambda)x_2^t = M(t)x_0 + \lambda[M(t)\int_0^t M(s)^{-1}B(s)u_1(s)ds] + (1 - \lambda)[M(t)\int_0^t M(s)^{-1}B(s)u_2(s)ds] = x(t)
\]

Thus \( (\lambda x_1^t + (1 - \lambda)x_2^t) \in \text{Acc}(x_0, t) \) and \( \text{Acc}(x_0, t) \) is convex.

\[\square\]

**Figure 2.6:** Reachable set in the linear case, from [26].

### A more simple case

Let us consider: \( r = 0 \) and \( x_0 = 0 \). The system becomes:

\[
x'(t) = A(t)x(t) + B(t)u(t) ; \quad x(0) = x_0
\]

and its (unique) solution writes:

\[
x(t) = M(t)\int_0^t M(s)^{-1}B(s)u(s)ds
\]

Thus \( x \) is linear in \( u \). Then we have

**Proposition 2.3.1.** Let us consider the system: \( x'(t) = A(t)x(t) + B(t)u(t) ; \ x(0) = x_0 \), and \( \Omega = \mathbb{R}^m \). Then for all \( t > 0 \), the reachable set \( \text{Acc}(0, t) \) is a vectorial subspace of \( \mathbb{R}^n \).

If we assume in addition \( B(t) = B \) constant, then for any \( 0 < t_1 < t_2 \), we have: \( \text{Acc}(0, t_1) \subset \text{Acc}(0, t_2) \).

We refer to [26] for the proof.
2.3.2 Controllability of autonomous linear systems: Kalman’s condition

In the following we consider a first order linear autonomous ODE system in \( \mathbb{R}^n \):
\[
x'(t) = Ax(t) + Bu(t) + r(t)
\]
with \( A \) and \( B \) independent of \( t \). Furthermore, we set \( \Omega = \mathbb{R}^m \) (i.e. we do not constraint the control variable).

We say that the system is controllable at any time \( T \) if \( \text{Acc}(x_0, T) = \mathbb{R}^n \); it means that for any \( x_0 \) and \( x_1 \) in \( \mathbb{R}^n \), there exists a control \( u \) such that \( x(0) = x_0 \) and \( x(T) = x_1 \).

**Theorem 2.3.2.** Under the assumptions above, the first order linear autonomous ODE system is controllable at any time \( T \) if and only if the rank of matrix \( C = (B, AB, \ldots, A^{n-1}B) \) is equal to \( n \).

The matrix \( C \) is called the Kalman matrix and the condition \( \text{rank}(C) = n \) is called the Kalman controllability condition.

Since the Kalman’s condition does not depend neither on \( T \) nor on \( x_0 \), then the first order linear autonomous system is controllable at any time from any initial condition. (R. Kalman is a hungarian-american mathematician-electrical engineer, born in 1930).

**Sketch of the proof.** First, one proves that \( \text{rank}(C) = n \) if and only if the following linear application \( \Phi \) is surjective.
\[
\Phi : L^\infty([0, T], \mathbb{R}^m) \mapsto \mathbb{R}^n ; \quad u \mapsto \int_0^T \exp((T-t)A)Bu(t)dt
\]

Thus \( \Phi : L^\infty([0, T], \mathbb{R}^m) = \mathbb{R}^n \). In other respect, given \( u \), we have:
\[
x(T) = \exp(TA)x_0 + \int_0^T \exp((T-t)A)(Bu(t) + r(t))dt
\]

Thus the reachable states is:
\[
\text{Acc}(x_0, T) = \exp(TA)x_0 + \int_0^T \exp((T-t)A)r(t)dt + \Phi(L^\infty) = \mathbb{R}^n
\]

Therefore, the system is controllable.

We refer to [bf] for the full proof of the theorem; also to your course of automatism [8].

These results give an overview of notions of controllability for linear systems. We refer e.g. to [26] for more details and the study of non-linear systems.
2.4 The Linear-Quadratic (LQ) problem

Let $A, B$ and $s$ be three mappings defined from $I = [0,T]$ into $M_{n,n}(\mathbb{R})$, $M_{n,m}(\mathbb{R})$ and $\mathbb{R}^n$ respectively. The three mappings are assumed to be bounded i.e. $L^\infty(I)$ (this assumption could be relaxed since locally integrable would be sufficient). We consider the following linear 1st order ODE.

\[
\begin{align*}
\text{Given } u(t), \text{ find } x(t) \text{ such that: } \\
x'(t) &= A(t)x(t) + B(t)u(t) + s(t) \text{ for } t \in I = [0,T] \\
\text{with the initial condition: } x(0) = x_0.
\end{align*}
\]

The function $u(t)$ is assumed to be in $L^\infty(I)$.

In other words, we consider a phenomena which can be modelled by this linear ODE. (2.1) is the direct model, $x(t)$ is the state of the system, and $u(t)$ will be the control of the system.

Existence and uniqueness of the solution. One knows (classical theorem of existence) that (2.1) has one and only one solution $x(t)$, $x(t)$ continuous from $I$ into $\mathbb{R}^n$.

Let us remark that one knows to give an explicit expression of $x$ in an integral form; the unique solution $x(t)$ depends on $u(t)$ of course.

A controllability problem reads as follows: given $x_1 \in \mathbb{R}^n$, is there exist a control $u(t)$ such that $x(t)$ goes from $x_0$ (at initial time) to $x_1$ in finite time ?

An optimal control problem reads as follows: is there exist a control $u(t)$ such that it minimizes a given criteria $j(u)$ ?

$j(u)$ is called the cost function. Potentially, the optimal control problem can be considered with the additional constraint: $x(t)$ must go from $x_0$ (initial state) to $x_1$ given in finite time.

The course [8] treats mainly controllability problems for linear systems. In the present course, we consider optimal control problems.

Remark 2.4.1. - Since the previous spring academic example is modelled by a linear ODE in the case $k_2 = 0$, the results which follow will apply to such a case.

- Among the historical optimal control problems, let us cite the so-called brachistochrone problem (end of the 17th century, from the greek words "brakhisto" (shorter) and "chronos" (time)) where the system is a simple 1d gravitational dynamic without friction, and $j$ is time, from $x_0$ to $x_1$.

Quadratic cost function

We consider the following ”quadratic” cost function $j(u)$. Quadratic cost are often use in practice and particularly in variational data assimilation (see next chapter). Somehow, such cost functions set the inverse problem as a least-square problem.

Let $Q$, $W$ and $U$ be given in $M_{n,n}(\mathbb{R})$, $M_{n,n}(\mathbb{R})$ and $M_{m,m}(\mathbb{R})$ respectively. We assume that each of them are symmetric positive, furthermore, $U$ is define. These three matrices define
corresponding semi-norms and norms respectively. We set:

\[ j(u) = \frac{1}{2} \int_0^T \| x(t) \|_W^2 \, dt + \frac{1}{2} \int_0^T \| u(t) \|_U^2 \, dt + \frac{1}{2} \| x(T) \|_Q^2 \]  

(2.2)

The three terms are the time averaged cost of the state, the control and the final state, respectively in the metrics \( W, U \) and \( Q \).

In fact, \( Q, W \) must be symmetric positive, but definite is not necessary. As a matter of fact, for example, one can have the cost function minimal for a vanishing control.

Since the definition of the cost function cost is quadratic, and the model is linear, the natural functional space for control variable is \( M = L^2([0,T], \mathbb{R}^m) \).

Let us point out that the a-priori natural space \( C^0([0,T], \mathbb{R}^m) \) is not a Hilbert space...

Let us recall that for a matrix \( M \) symmetric positive definite, in vertu of Courant-Fischer’s theorem (Rayleigh’s quotient), there exists a constant \( c > 0 \) such that:

\[ \forall t \in [0, T], \ \forall v \in \mathbb{R}^m, \ \| v \|^2_M \geq c \| v \|^2 \]

In other words, the (linear) operator \( M \) is coercive, uniformly in time.

The linear-quadratic optimal control problem we consider reads as follows: Given \( x_0 \) and \( T \), find \( u(t) \) such that \( x(t) \), solution of (2.1), minimizes the cost function \( j(u) \) defined by (2.2).

The linear-quadratic problem seems a-priori idealistic but not so much. Indeed, cost functions are often defined quadratic and many ”simple” automatism process may be modelled by a linear equation. Furthermore, for more complex non-linear models, if linearized around a given state, it is by definition locally linear...

In addition, one knows to write a lot of very useful and instructive properties for the LQ problem, both in a mathematical and numerical point of view.

For a sake of simplicity, in the sequel we consider the source term \( s(t) = 0 \) in the direct model.

### 2.5 Necessary conditions: Pontryagin’s maximum principle

#### 2.5.1 Existence and uniqueness of the solution

We have the following result of existence and uniqueness.

**Theorem 2.5.1.** There exists an unique solution \( u \in M \) which minimize \( j(u) \) defined by (2.2) with the ”constraint” \( x(t) \) solution of (2.1).

In other words, there exists an unique optimal trajectory \( x(t) \) to the LQ problem.
Proof. A) Existence. It is based on the convergence of minimizing sequence (calculus of variations, D. Hilbert, 1900 approx.).

Step 1) The cost function is bounded below: \( \inf \{ j(u), u \in M \} > -\infty \) since \( j(u) \geq 0 \). There exists a minimizing sequence \((u_n)\) defined for all \( t \in [0,T] \); i.e. a sequence such that:

\[
\lim_{n \to +\infty} j(u_n) = \inf \{ j(u), u \in M \}
\]

(As a matter of fact, \( \forall n \in \mathbb{N}, \exists v_n \) such that: \( m \leq j(v_n) < m + \frac{1}{n} \).

Step 2) There exists \( \alpha > 0 \) such that: \( j(u) \geq \alpha \|u\|^2_M \). Thus, the minimizing sequence \((u_n)\) is bounded in \( M \). Hence there exists a sub-sequence \((u_{n_k})\) which converges weakly to a control \( u \) in \( M \):

\[ u_{n_k} \to u \text{ in } L^2(I) \]

Step 3) Let us denote by \( x_n \) (resp. \( x \)) the state associated to \( u_n \) (resp. \( u \)). The system (2.1) is a first order linear O.D. E. (and with \( s(t) = 0 \)); one knows an explicit expression of the solution:

\[
\forall t, \quad x_n(t) = M(t)x_0 + M(t) \int_0^t M(s)^{-1}B(s)u_n(s)ds \quad (2.3)
\]

with \( M(t) \in M_{n,n}(\mathbb{R}) \) such that: \( M'(t) = A(t)M(t), \quad M(0) = Id. \) (If \( A(t) = A \) constant then \( M(t) = \exp(tA) \)).

Similarly, we have: \( \forall t, \quad x(t) = M(t)x_0 + M(t) \int_0^t M(s)^{-1}B(s)u(s)ds \). Thus, we obtain that the sequence \((x_{n_k})\) converge to \( x \).

Passing to the limit in (2.3), we obtain the existence of \( x_u \), solution corresponding to \( u \).

Step 4) It remains to prove that \( u \) minimizes \( j \). Since \( u_n \to u \) in \( L^2 \), since \( j \) is continuous hence lower semi-continuous, we have (by definition):

\[
j(u) \leq \liminf_{n} j(u_n)
\]

and necessarily \( j(u) = \inf_{v \in M} j(v) \).

Figure 2.7: A function \( j \) lower semi-continuous at \( u_0 \): for \( u \) close to \( u_0 \), \( j(u) \) is either close to \( u_0 \) or lower than \( u_0 \).

In other words, \( u(t) \) minimizes \( j(u) \) (\( j(u) = \min_{v \in M} j(v) \)) and the corresponding state (trajectory) \( x_u \) is an optimal trajectory.
B) Uniqueness. Let us prove that \( j(u) \) is strictly convex i.e. 
\[
\forall (u_1, u_2) \in M^2, \forall t \in [0, 1[, \quad j(tu_1 + (1 - t)u_2) < tj(u_1) + (1 - t)j(u_2) \text{ unless } u_1 = u_2.
\]
For all \( t \), \( \|u(t)\|_U \) is a norm hence convex and not strictly convex (proof: triangle inequality). But in a Hilbert space, the square of a norm (eg \( \|u(t)\|^2_U \) ) is strictly convex (see e.g. [3] chap. 10, p118).

Let us define the "mapping model":
\[
\mathcal{M} : u(t) \mapsto x_u(t); \quad M = L^2(I, \mathbb{R}^m) \to C^0(I, \mathbb{R}^n)
\]

We have:

**Lemma 2.5.1.** The mapping model \( \mathcal{M}(u(t)) \) is affine, thus convex for all \( t \) in \([0, T]\).

The proof is similar to those of Theorem 2.3.1. \( \square \)

**Exercice 2.5.1.** Prove this lemma.

**Proof of uniqueness (continued).**

In virtue of Lemma 2.5.1, and since \( \|\cdot\|_W \) and \( \|\cdot\|_Q \) are semi-norms hence convex, the cost function \( j(u) \) is strictly convex. Finally, the uniqueness follows straightforwardly.

Let \( u_1 \) and \( u_2 \) be such that: \( j(u_k) = \inf_{v \in M} j(v) \), \( k = 1, 2 \). We have: \( j(tu_1 + (1 - t)u_2) < tj(u_1) + (1 - t)j(u_2) \).

Hence: \( j(tu_1 + (1 - t)u_2) < \inf_{v \in M} j(v) \) unless \( u_1 = u_2 \), which must be the case. \( \square \)

**Remark 2.5.1.** In the autonomous case (\( A \) and \( B \) constant), we have:
\[
\|x'(t)\| \leq \|A\|\|x(t)\| + \|B\|\|u(t)\| \leq \text{cst}(\|x(t)\|^2 + \|u(t)\|^2)
\]

Then, if all hypothesis are satisfied in \( I = [0, +\infty) \) then \( x'(t) \) is in \( L^1(I) \) and necessarily the minimizing trajectory \( x(t) \) tends to 0 when \( t \) tends to \(+\infty\).

## 2.5.2 Pontryagin’s maximum principle in the LQ case

We have the following Pontryagin’s maximum principle in the LQ case:

**Theorem 2.5.2.** The trajectory \( x(t) \) associated with the control \( u(t) \), is optimal for the LQ problem (system (2.1) (2.2)) if there exists an adjoint vector \( p(t) \) which satisfies:
\[
p'(t) = -p(t)A(t) + x(t)^TW(t) \text{ for almost } t \in [0, T]
\]
with the final condition: \( p(T) = -x(T)^TQ \).

Furthermore, the optimal control \( u \) satisfies:
\[
u(t) = U(t)^{-1}B(t)^Tp(t)^T \text{ for almost } t \in [0, T]
\]
The ideas of the proof based on ‘calculus of variations’ is similar to those of Theorem 3.8.1 in next chapter. Since we will detail the latter in next chapter (non-linear elliptic PDE case), we refer to [26] for the present proof.

\[ \text{Remark 2.5.2. In the general case, i.e. a non-linear state equation hence a non convex cost function, the Pontryagin’s maximum principle is a necessary condition only and its proof is quite complex. It makes introduce the so-called hamiltonian of the system and its maximisation (thus the terminology), see e.g. [26]. (L. Pontryagin, 1908-1988; W. Hamilton, Irish physicist, 1805-1865). In the LQ case, the Pontryagin’s maximum principle is a necessary and sufficient condition.} \]

Remark 2.5.3.
- In case of an infinite time interval \( T = +\infty \), the final condition becomes: \( \lim_{t \to +\infty} p(t) = 0 \).
- If the system has a source term \( s(t) \) non null, the result remains true.

2.5.3 The Hamiltonian

Let us present the Hamiltonian \( H \) in the present LQ case. It writes:

\[
H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \\
H(x, p, u) = p(Ax + Bu) - \frac{1}{2}(x^T W x + u^T U u)
\]

(Recall: \( \|x\|_N^2 = x^T N x \)).

Then, the necessary and sufficient equations of Theorem 2.5.2 re-writes as follows:

\[
\begin{align*}
\dot{x}(t) &= \partial_p H(x, p, u) = Ax + Bu \\
\dot{p}(t) &= \partial_x H(x, p, u) = pA + x^T W \\
\partial_u H(x, p, u) &= 0 \quad \text{since } pB - u^T U = 0
\end{align*}
\]

where \( p \) is considered to be a line vector in \( \mathbb{R}^n \).

In other words, a necessary and sufficient condition to solve the LQ problem, is the Hamiltonian must be maximized.

2.5.4 Example

Let us consider the control of the 1d trajectory of a vehicle (without friction forces). We have the simple scalar linear second order model:

\[
x''(t) = u(t) ; \quad x(0) = x'(0) = 0
\]

Given the final time \( T \), we seek to maximize the distance travelled with a minimum of energy. To this end, we consider the cost function:

\[
j(u) = -\alpha_1 x(T)^2 + \alpha_2 \int_0^T u^2(t) dt
\]
J. Monnier

with $\alpha_*$ given weight parameters, modelling the importance we want to give to each term of the cost function. For a sake of simplicity, we set: $\alpha_1 = \alpha_2 = 1$.

**Exercice 2.5.2.** Give the expression of the optimal control.

### 2.6 Feedback and Riccati’s equation

Riccati’s family (father and son), Italian mathematicians, 18th century.

In the LQ case, it is possible to write the control as a feedback (closed-loop control), and compute it by solving the Riccati’s equation. It is the purpose of the present section.

#### 2.6.1 Solution in the LQ case: Feedback and Riccati’s equation

We have

**Theorem 2.6.1.** Under the assumptions of the existence - uniqueness Theorem 2.3, the (unique) optimal control $u$ writes as a feedback function (closed-loop control):

$$u(t) = K(t)x(t) \text{ with } K(t) = U(t)^{-1}B^T(t)E(t)$$

where the matrix $E(t)$, $E(t) \in M_n(\mathbb{R})$, is solution of the Riccati equation:

$$E'(t) = W(t) - A^T(t)E(t) - E(t)A(t) - E(t)B(t)U^{-1}(t)B^T(t)E(t) \quad \forall t \in (0, T)$$

with the final condition: $E(T) = -Q$.

For all $t$, the matrix $E(t)$ is symmetric. Furthermore, since $Q$ and $W$ are positive definite, $E(t)$ is positive definite too.

We refer to [26] for the proof.

#### 2.6.2 And in general cases?

In practice, optimal control problems are often non-linear, thus the results presented for the LQ problem do not apply directly, in particular the analytic expression of the optimal control...

Nevertheless, a good understanding of the LQ solution structure (e.g. the strict convexity of the resulting cost function) is useful to tackle non-linear problems.

Then, to solve a non-linear optimal control problem (in the sense compute an optimal control $u$ and the optimal trajectory $x^*$), one can use numerical methods to solve the necessary first-order optimality conditions based on the Hamiltonian system.

For Partial Derivatives Equations (PDE) systems, the Pontryagin’s maximum principle does not apply anymore; the Riccati’s equations are generally not known. These questions are still
an active mathematical research field and some recent results exist (e.g. the Riccati’s equations for the Navier-Stokes fluid flows equations at low Reynolds number etc).

Nevertheless, for non-linear PDEs one can obtain easily write equations characterizing the optimal control solution: it is the \textit{optimality system} based on the \textit{adjoint equations}. In next section, we study this approach for non-linear elliptic PDEs systems.