Ensemble Methods for Data Assimilation 2022-2023

Olivier THUAL and Mohamed SAADI
Toulouse INP-ENSEEIHT/MFEE

Course:
Mathematical Methods for Data Exploitation (MMED)

Teaching Unit:
Big Data for Geosciences

Where are we? MFEE/SEE-BD and MFEE/MSN-BD

Teaching Units: Big Data for Geosciences (5 ECTS)

- Course : Use of Artificial Intelligence in Forecasting (50%)
- Course : Mathematical Methods of Data Exploitation (50%)

Course : Mathematical Methods for Data Exploitation (MMED)

1. Part 1: Uncertainty Quantification, H. ROUX
2. Part 2: Ensemble Methods for Data Assimilation, O. THUAL and M. SAADI
3. Exam: Project presentations, H. ROUX, O. THUAL and M. SAADI

How will the course be evaluated

Report containing two parts

1. Part 1: Uncertainty Quantification
2. Part 2: Ensemble Methods for Data Assimilation

Report for Part 2

- About 10 pages of results
- Sources of the developed post-processing programs

Oral presentation

- 8 mn each
- Choice between Part 1 or Part 2

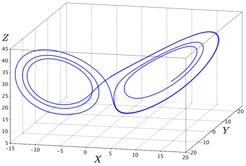
What is the program of the course?

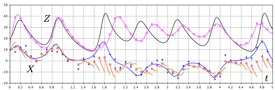
CM 1	Course presentation ① ENGINEER APPLICATIONS ② INCREMENTAL METHOD ③ TIME DEPENDENT MODELS
TDM1	Improvement of Homework A
CM2	◯ Deadline for Homework A ④ ENSEMBLE METHODS IN $\mathbb{R}$ ⑤ ENSEMBLE METHODS IN $\mathbb{R}^N$
TDM2	Programming Homework B
TDM3	Improvement of Homework B
Exam	◯ Deadline for Homework B Oral presentation of Part 1 or Part 2 reports

Course notes

Olivier THUAL

Data assimilation for engineers





Toulouse INP - ENSEEIHT

"Fluid Mechanics, Energetics and Environment" Department

Year 2022-2023, November 23, 2022

Contents

Outlines of the slides

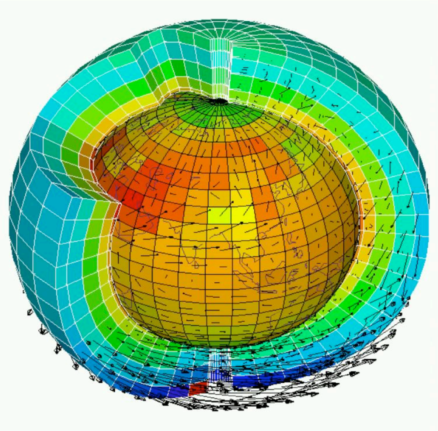
1. ENGINEER APPLICATIONS
2. INCREMENTAL METHODS $\rightarrow$ Homework A (Ex. 2.1)
3. TIME DEPENDENT MODELS $\rightarrow$ Homework A (Ex. 3.1)
4. ENSEMBLE METHODS IN $\boldsymbol{R}$ $\rightarrow$ Homework B (4.2.2 and 4.2.3)
5. ENSEMBLE METHODS IN $\boldsymbol{R}^N$ $\rightarrow$ Homework B (4.2.4)

1. ENGINEER APPLICATIONS

Weater forecast and other applications

Data assimilation is part of the modelling science for researcheres and engineers.

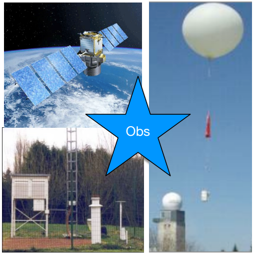
Data assimilation for weather forecast

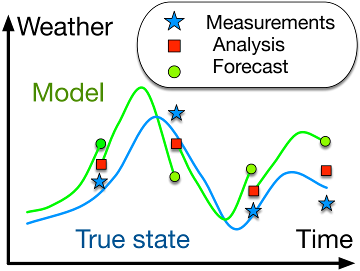


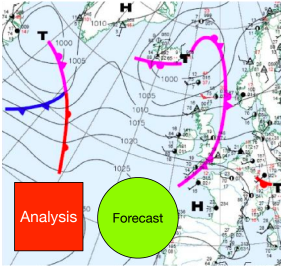
Atmospheric model:

Fluid mechanics equations for winds, pressure, temperature and humidity.

Data assimilation chain : a minimization problem







$$J(\blacksquare) = \frac{1}{2}\| \blacksquare - \bullet \|_B^2 + \frac{1}{2}\| \bigstar - \mathcal{G}(\blacksquare)\|_R^2$$

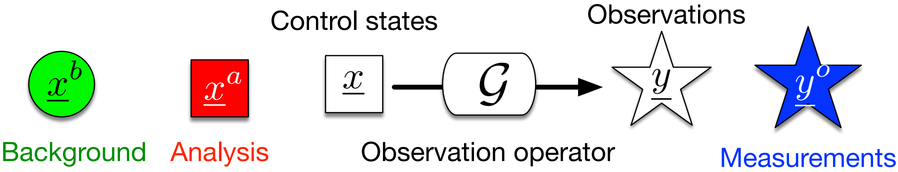
Looking for the present weather : the analysis

A state close to both the previous forecast and the field measurments

The standard formalism of data assimilation

The analysis minimizes a cost function

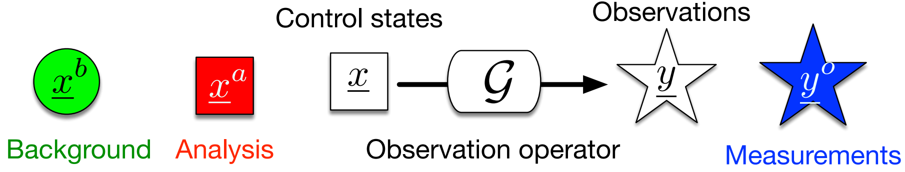
$$J(\underline{x}) = \frac{1}{2}\left(\underline{x} - \underline{x}^b\right)^T \underline{\underline{B}}^{-1}\left(\underline{x} - \underline{x}^b\right) + \frac{1}{2}\left[\underline{y}^o - \mathcal{G}(\underline{x})\right]^T \underline{\underline{R}}^{-1}\left[\underline{y}^o - \mathcal{G}(\underline{x})\right]$$



The metrics depends on the uncertainties

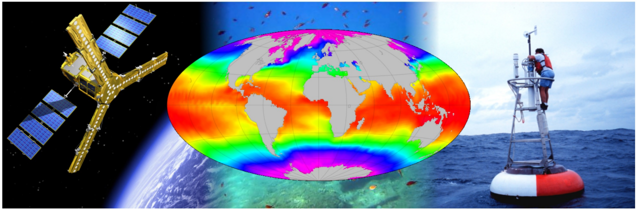
- The $N \times N$ “background error covariance matrix” $\underline{\underline{B}}$
- The $M \times M$ “observation error covariance matrix” $\underline{\underline{R}}$

Data assimilation for meteorology



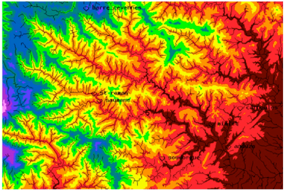
Control space	Observation operator	Observation space
• 3D fields: T, P, H, U, V • Order ten millions of grid points	Evolution model of the primitive equations of the atmosphere	• Satellite data • In-situ data • Order one million of observations

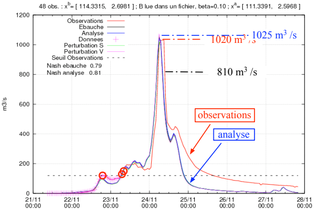
Data assimilation for oceanography



Control space	Observation operator	Observation space
• 3D fields: T, P, S, U, V • 2D fields: sea surface level • Order ten millions of grid points	Evolution model of the primitive equations of the ocean	• Satelitte data: sea surface temperature, altimetry • In-situ data: temperature, salinity... • Order thousands of observations

Data assimilation for hydrology





Control space	Observation operator	Observation space
• 1D fields: h, U, T • Model parameters: friction, soil water content... • Order thousands of grid points	Shallow water (Saint-Venant) equations	• Satelitte data: altimetry • In-situ data: water height, piezometric height • Order hundred observations

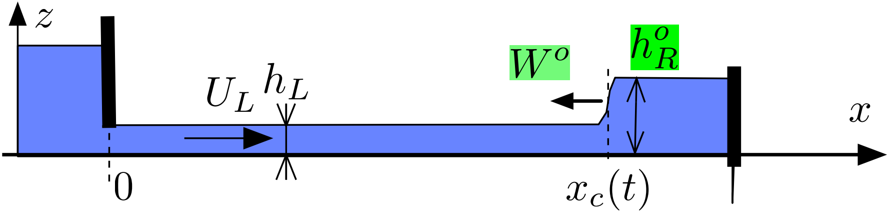
2. INCREMENTAL METHOD

Linearization of the observation operator

How to compute the minimum of a nonlinear cost function through a linear approximation of the observation operator around the background state.

How will the bore propagate?

A very simple geophysical model



Mass conservation:

$$h_L(U_L - W) = h_R(U_R - W) \quad \Longrightarrow W = \frac{-h_L\, U_L}{h_R - h_L}$$

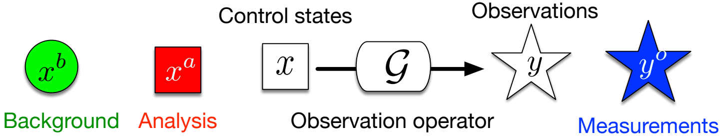
Notations $\quad x \in \boldsymbol{R}, \quad \mathcal{G} : \boldsymbol{R} \to \boldsymbol{R}, \quad y \in \boldsymbol{R}$:

$$x = h_R\ , \quad y = W\ , \quad \text{and} \quad y = \mathcal{G}(x) = \frac{-h_L\, U_L}{x - h_L} = \frac{-q}{x - h_L}$$

Best estimate of the height $x = h_R$

Too much information in: $x = h_R, \quad \mathcal{G}(x) = \frac{-q}{x - h_L}, \quad y = W$

- We know approximatively $x = x^b$ with an incertainty error σ_b
- We know approximatively $y = y^o$ with an incertainty error σ_r
- What is the best estimate x^a of x knowing that $y = \mathcal{G}(x)$?

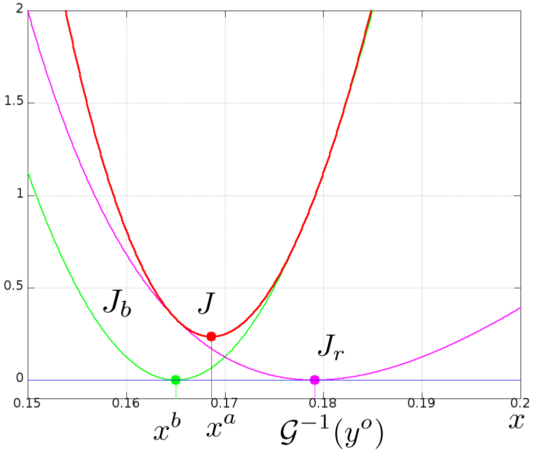


The analysis x^a minimizes the cost function:

$$J(x) = \frac{(x - x^b)^2}{2\,\sigma_b^2} + \frac{[y^o - \mathcal{G}(x)]^2}{2\,\sigma_r^2}$$

Plotting the cost function $J(x)$

$$J(x) = \frac{(x - x^b)^2}{2\,\sigma_b^2} + \frac{[y^o - \mathcal{G}(x)]^2}{2\,\sigma_r^2} = J_b(x) + J_r(x)$$



$$J_b(x) = \frac{(x - x^b)^2}{2\,\sigma_b^2}$$

$$J_r(x) = \frac{[y^o - \mathcal{G}(x)]^2}{2\,\sigma_r^2}$$

with $\quad \mathcal{G}(x) = \dfrac{-q}{x - h_L}$

Incremental cost function

The cost function to minimize

$$J(x) = \frac{(x - x^b)^2}{2\,\sigma_r^2} + \frac{[y^o - \mathcal{G}(x)]^2}{2\,\sigma_r^2} \qquad \text{with} \qquad \mathcal{G}(x) = \frac{-q}{x - h_L}$$

Linearization of $\mathcal{G}$ around x^b:

$$\mathcal{G}(x) = \mathcal{G}(x^b + \delta x) \sim \mathcal{G}(x^b) + G\,\delta x \qquad \text{with } \delta x = x - x^b$$

One can compute $G = \mathcal{G}'(x^b) = q/(x^b - h_L)^2$

The incremental cost function J_{inc} is an approximation of J:

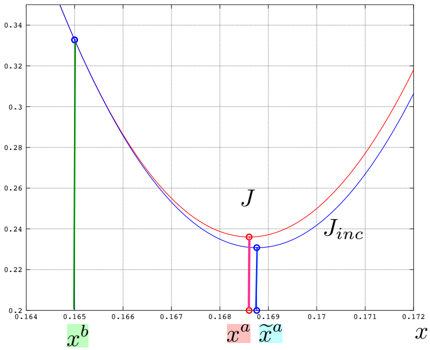
$$J_{inc}(x^b + \delta x) = \frac{(\delta x)^2}{2\,\sigma_b^2} + \frac{(d - G\,\delta x)^2}{2\,\sigma_r^2}$$

where $d = y^o - \mathcal{G}(x^b)$ is the "innovation"

Plotting the incremental cost function J_{inc}

Gradient of the function $J_{inc}(x^b + \delta x) = \frac{(\delta x)^2}{2\,\sigma_b^2} + \frac{(d - G\,\delta x)^2}{2\,\sigma_r^2}$:

$$J'_{inc}(x^b + \delta x) = \frac{\delta x}{\sigma_b^2} + G\,\frac{G\,\delta x - d}{\sigma_r^2}$$



J'_{inc} vanishes for: $\widetilde{\underline{x}}^a = \underline{x}^b + \widetilde{\delta x}^a$

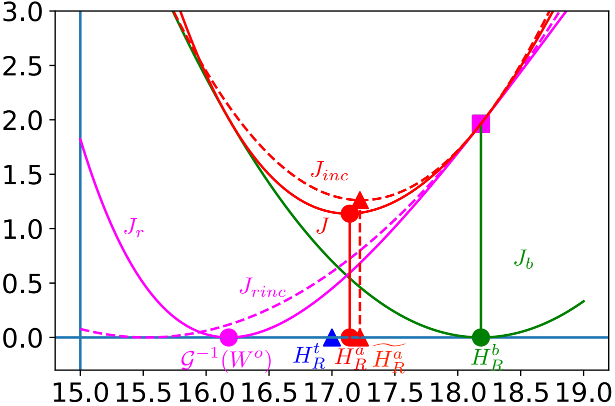
with $\widetilde{\delta x}^a = K\,d$ where:

Innovation: $d = y^o - \mathcal{G}(x_b)$

Gain: $K = \left(\frac{1}{\sigma_b^2} + \frac{G^2}{\sigma_r^2} \right)^{-1} \frac{G}{\sigma_r^2}$

or $K = \sigma_b^2\,G\,\left(G^2\,\sigma_b^2 + \sigma_r^2 \right)^{-1}$

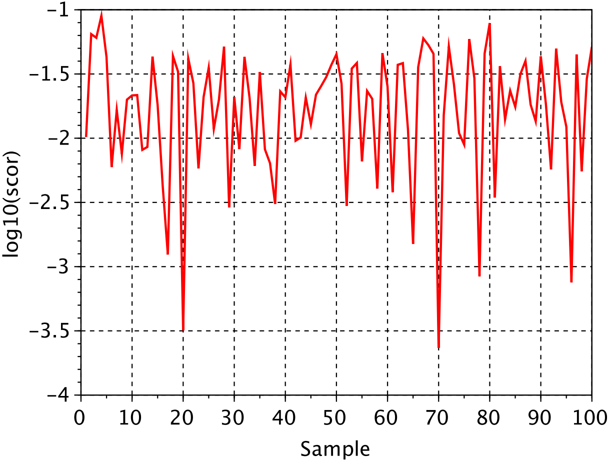
Validity of the incremental method on the bore example



Blocked height $x = H_R$ and bore velocity $y = W$

$$\mathcal{G}(x) = \frac{-q}{x - h_L} \quad \text{and} \quad G = \mathcal{G}'(x^b) = \frac{q}{(x - h_L)^2}$$

Score of the incremental method for the bore experiment

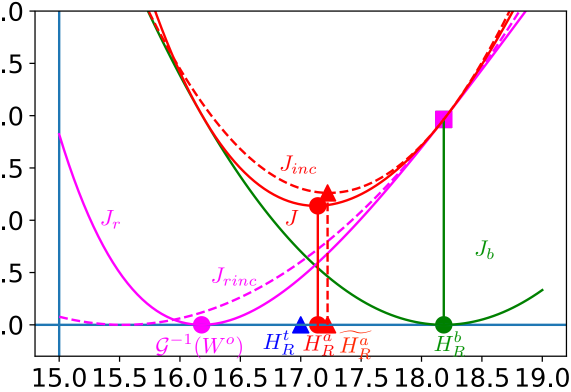


Experimental values

$q = 7,\ h_L = 5\ h_R = x^t = 17,\ x^b = 18,\ \sigma_b = 1,\ \sigma_r = .03$

Homework A (1/2). Exo 2.1 : "How will the bore propagate?"

Hands-on from: `https://olivier-thual.fr/130202`



$$\mathcal{G}(x) = \frac{-q}{x - h_L}$$

$$G = \mathcal{G}'(x^b) = \frac{q}{(x - h_L)^2}$$

1. Read program.
2. Launch program
3. Compute scores
4. Replace $\mathcal{G}$
5. Count improvements

Basic linear algebra

Vectors are $1 \times N$ or $1 \times M$ matrices

$$\underline{x} = \begin{pmatrix} x_1 \\ \dots \\ x_j \\ \dots \\ x_N \end{pmatrix} , \quad \underline{y} = \begin{pmatrix} y_1 \\ \dots \\ y_i \\ \dots \\ y_M \end{pmatrix} , \quad \begin{cases} \underline{x}^T = (x_1, \dots, x_j, \dots, x_N) \\ \\ \underline{y}^T = (y_1, \dots, y_i, \dots, y_M) \end{cases}$$

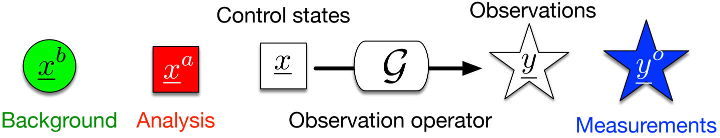
Example of a $M \times N$ matrix considered as a linear operator:

$$\underline{y} = \underline{\underline{H}}\, \underline{x} \quad \Longleftrightarrow \quad \begin{pmatrix} y_1 \\ \dots \\ y_i \\ \dots \\ y_M \end{pmatrix} = \begin{pmatrix} H_{11} & \dots & H_{1j} & \dots & H_{1N} \\ \dots & \dots & \dots & \dots & \dots \\ H_{i1} & \dots & H_{ij} & \dots & H_{iN} \\ \dots & \dots & \dots & \dots & \dots \\ H_{M1} & \dots & H_{Mj} & \dots & H_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_j \\ \dots \\ x_N \end{pmatrix}$$

Generic cost function for data assimilation

Searching the analysis $\underline{x}^a$ that minimize the cost function

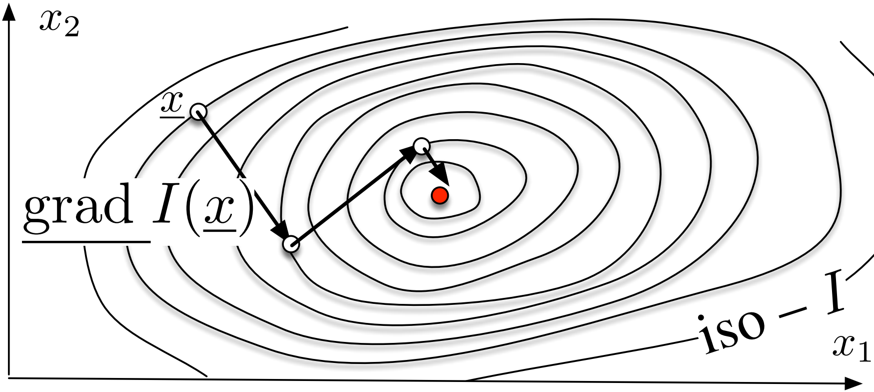
$$J(\underline{x}) = \frac{1}{2}\left(\underline{x}-\underline{x}^b\right)^T \underline{\underline{B}}^{-1}\left(\underline{x}-\underline{x}^b\right) + \frac{1}{2}\left[\underline{y}^o - \mathcal{G}(\underline{x})\right]^T \underline{\underline{R}}^{-1}\left[\underline{y}^o - \mathcal{G}(\underline{x})\right]$$



- Control space $\underline{x} \in \boldsymbol{R}^N$ and Observation space $\underline{y} \in \boldsymbol{R}^M$
- Background $\underline{x}^b$ known with errors $\underline{\epsilon}^b$
- Measurement $\underline{y}^o$ known with errors $\underline{\epsilon}^o$
- Covariance background error matrix $\underline{\underline{B}}$ with $B_{ij} = \left\langle \epsilon_i^b \, \epsilon_j^b \right\rangle$
- Covariance observation error matrix $\underline{\underline{R}}$ with $R_{ij} = \left\langle \epsilon_i^o \, \epsilon_j^o \right\rangle$

Gradient of a scalar function

$$\underline{\text{grad}}\ I(\underline{x}) = \left(\frac{\partial I}{\partial x_1},\ \frac{\partial I}{\partial x_2},\ \cdots \frac{\partial I}{\partial x_j},\ \cdots \cdot \frac{\partial I}{\partial x_N} \right)^T$$



Examples of gradient computations

$\mathbf{J}(\underline{\mathbf{x}})$	$\underline{\text{grad}}\ \mathbf{J}(\underline{\mathbf{x}})$
$\underline{u}^T\ \underline{x}$	$\underline{u}$
$\underline{u}^T\ \underline{\underline{M}}\ \underline{x}$	$\underline{\underline{M}}^T\ \underline{u}$
$\underline{x}^T\ \underline{\underline{M}}\ \underline{x}$	$(\underline{\underline{M}} + \underline{\underline{M}}^T)\ \underline{x}$
$\underline{x}^T\ \underline{\underline{H}}^T\ \underline{\underline{S}}\ \underline{\underline{H}}\ \underline{x}$	$\underline{\underline{H}}\ (\underline{\underline{S}} + \underline{\underline{S}}^T)\ \underline{\underline{H}}^T\ \underline{x}$

Incremental cost function through a linearization of $\mathcal{G}$

$$J(\underline{x}) = \frac{1}{2}\left(\underline{x} - \underline{x}^b\right)^T \underline{\underline{B}}^{-1}\left(\underline{x} - \underline{x}^b\right) + \frac{1}{2}\left[\underline{y}^o - \mathcal{G}(\underline{x})\right]^T \underline{\underline{R}}^{-1}\left[\underline{y}^o - \mathcal{G}(\underline{x})\right]$$

Linearization of $\mathcal{G}$ around the background $\underline{x}^b$:

$$\mathcal{G}(\underline{x}^b + \underline{\delta x}) \approx \mathcal{G}(\underline{x}^b) + \underline{\underline{G}}\,\underline{\delta x}$$

Incremental cost function

$$J_{inc}(\underline{x}^b + \underline{\delta x}) = \frac{1}{2}\underline{\delta x}^T\,\underline{\underline{B}}^{-1}\,\underline{\delta x} + \frac{1}{2}\left(\underline{d} - \underline{\underline{G}}\,\underline{\delta x}\right)^T\,\underline{\underline{R}}^{-1}\,\left(\underline{d} - \underline{\underline{G}}\,\underline{\delta x}\right)$$

where $\underline{d} = \underline{y}^o - \mathcal{G}(\underline{x}^b)$ is the innovation vector

Minimum of the incremental cost function

Knowing the innovation vector $\underline{d} = \underline{y}^o - \mathcal{G}(\underline{x}^b)$:

$$J_{inc}(\underline{x}^b + \underline{\delta x}) = \frac{1}{2}\underline{\delta x}^T \, \underline{\underline{B}}^{-1} \, \underline{\delta x} + \frac{1}{2}\left(\underline{d} - \underline{\underline{G}}\,\underline{\delta x}\right)^T \, \underline{\underline{R}}^{-1} \, \left(\underline{d} - \underline{\underline{G}}\,\underline{\delta x}\right)$$

Gradient of the incremental cost function

$$\text{g}\underline{\text{rad}} \; J_{inc}(\underline{x}^b + \underline{\delta x}) = \underline{\underline{B}}^{-1}\,\underline{\delta x} - \underline{\underline{G}}^T\,\underline{\underline{R}}^{-1}\,\left[\underline{d} - \underline{\underline{G}}\,\underline{\delta x}\right]$$

The minimum $\underline{\widetilde{x}}^a$ is found through $\text{g}\underline{\text{rad}} \; J_{inc}(\underline{x}^a) = \underline{0}$:

$$\underline{\widetilde{x}}^a = \underline{x}^b + \underline{\underline{K}}\,\underline{d}$$

where the gain matrix is: $\underline{\underline{K}} = \left(\underline{\underline{B}}^{-1} + \underline{\underline{G}}^T\,\underline{\underline{R}}^{-1}\,\underline{\underline{G}}\right)^{-1}\,\underline{\underline{G}}^T\,\underline{\underline{R}}^{-1}$

and is also equal to (SMW identity): $\underline{\underline{K}} = \underline{\underline{B}}\,\underline{\underline{G}}^T\,\left(\underline{\underline{G}}\,\underline{\underline{B}}\,\underline{\underline{G}}^T + \underline{\underline{R}}\right)^{-1}$

5. TIME DEPENDENT MODELS

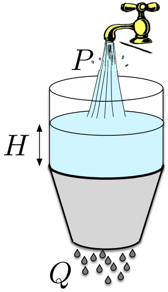
Time evolution of a system

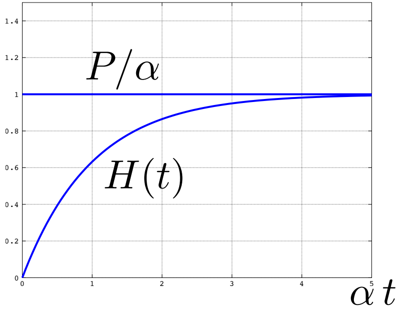
Data are available at different times and the model is a dynamical system.

Will the water overflow?

Reservoir model

$$\frac{dH(t)}{dt} = -\alpha\, H(t) + P \qquad \text{with} \qquad H(0) = 0$$





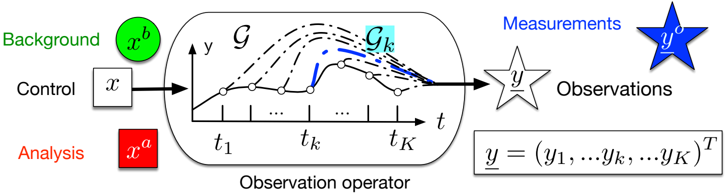
Exact solution

$$H(t) = (P/\alpha)\,[1 - \exp(-\alpha\, t)]$$

Assimilation of α

Measurements of $y_k = H(t_k)$ to determine $x = \alpha$

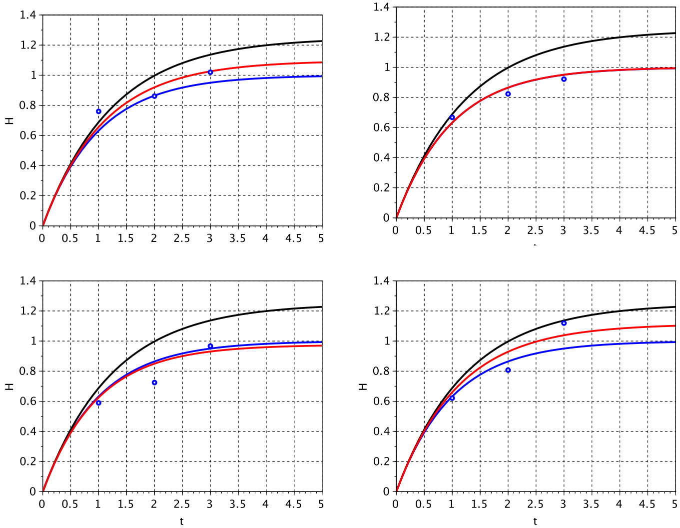
- $\underline{y}^o = (y_1^o, ...y_k^o, ...y_K^o)^T$
- $\mathcal{G} = (\mathcal{G}_1^T, ...\mathcal{G}_k^T, ...\mathcal{G}_K^T)^T$ with $\mathcal{G}_k(\alpha) = (1/\alpha)[1 - \exp(-\alpha\, t_k)]$



Cost function

$$J(\alpha) = \frac{1}{2}\frac{(\alpha - \alpha^b)^2}{\sigma_b^2} + \sum_{k=1}^{K} \frac{[y_k^o - \mathcal{G}_k(\alpha)^2]}{2\,\sigma_r^2}$$

Incremental method for the $x = \alpha$ reservoir example

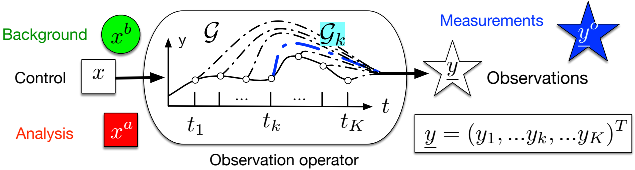


Red: analysis, Blue:true state, Black: background

Assimilation of (α, P)

Measurements of $y_k = H(t_k)$ to determine $\underline{x} = (\alpha, P)$

- $\underline{y}^o = (y_1^o, ...y_k^o, ...y_K^o)^T$
- $\mathcal{G} = (\mathcal{G}_1^T, ...\mathcal{G}_k^T, ...\mathcal{G}_K^T)^T$ with $\mathcal{G}_k(\alpha, P) = (P/\alpha)[1 - \exp(-\alpha\, t_k)]$



Cost function:

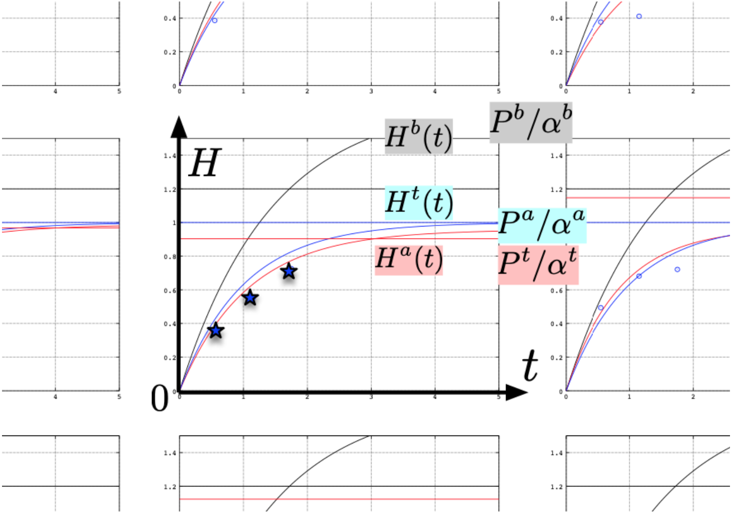
$$J(\alpha, P) = \frac{1}{2}(\alpha - \alpha^b, P - P^b)\, \underline{\underline{B}}^{-1} \begin{pmatrix} \alpha - \alpha^b \\ P - P^b \end{pmatrix}$$

$$+ \sum_{k=1}^{K} \frac{[y_k^o - \mathcal{G}_k(\alpha, P)]^2}{2\,\sigma_r^2}$$

Covariance background error matrix:

$$\underline{\underline{B}} = \begin{pmatrix} \sigma_\alpha^2 & \rho\,\sigma_\alpha\,\sigma_P \\ \rho\,\sigma_\alpha\,\sigma_P & \sigma_P^2 \end{pmatrix}$$

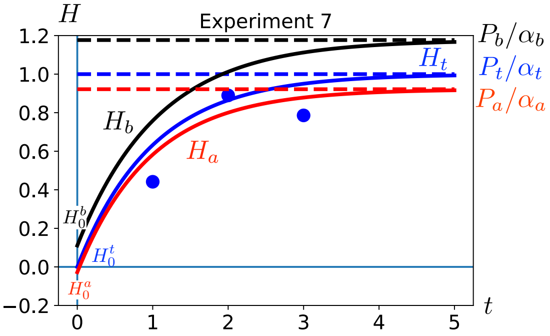
Incremental method for the assimilation of (α, P)



Red: analysis, Blue:true state, Black: background

Homework A (2/2). Exo3.1 : "Will the water overflow?"

Hands-on from: `https://olivier-thual.fr/130202`



1. Read the content of the program
2. Launch the program with the default parameters
3. Describe the program algorithm briefly
4. Change parameters and describe results
5. Enrich the program to compute scores

Homework A : Two exercices

Two exercices from the course notes :

- Exo 2.1 : "How will the bore propagate?"
- Exo3.1 : "Will the water overflow?"

Requirements

- A written report : 5 pages minimum
- The sources of the python program(s) or the Jupyter notebook(s)
- On Moodle before the dealine

4. ENSEMBLE METHODS IN *R*

Monte-carlo experiments

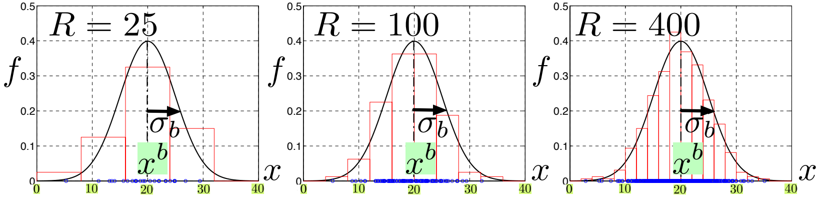
Use a great number of experiments :

- To compute scores
- To approximate derivatives

Gaussian random variable

Probability density function of a gaussian random variable:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma_b} \exp\left[-\frac{(x-x^b)^2}{2\,\sigma_b^2}\right] \quad \Longrightarrow \quad \langle \Phi(x) \rangle = \int_{-\infty}^{\infty} \Phi(x)\, f(x)\, dx$$

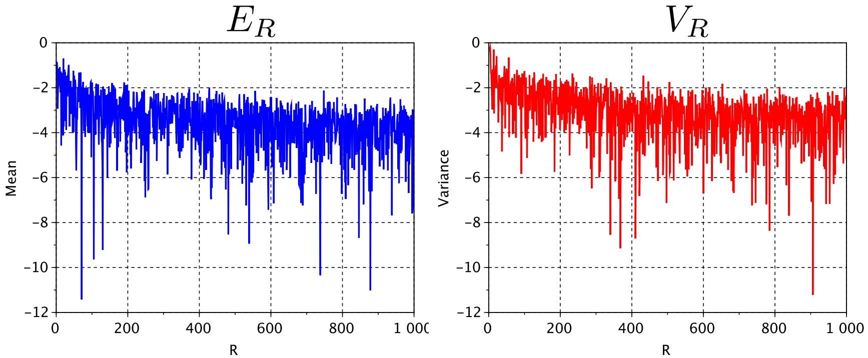


Mean $x^b = < x >$ Variance $\sigma_b^2 = \left\langle (x - x^b)^2 \right\rangle$ estimators:

$$E_R = \frac{1}{R} \sum_{r=1}^{R} x^{(r)} \sim x^b \text{ and } V_R = \frac{1}{R-1} \sum_{r=1}^{R} \left(x^{(r)} - E_R \right)^2 \sim \sigma_b^2$$

where $x^{(r)}$ for $r = 1, \dots R$ be R are draws of the random variable

Convergence of the estimators



Scores $\log_{10}|E_R - x^b|$ and $\log_{10}|V_R - \sigma_b^2|$:

$$E_R = \frac{1}{R}\sum_{r=1}^{R} x^{(r)} \sim x^b \text{ and } V_R = \frac{1}{R-1}\sum_{r=1}^{R}\left(x^{(r)} - E_R\right)^2 \sim \sigma_b^2$$

Computation of the gain with an ensemble method

We want to compute the gain K:

$K = \sigma_b^2\, G\, \left(G^2\, \sigma_b^2 + \sigma_r^2\right)^{-1}$ where $G = \mathcal{G}'(x^b)$ ... without computing G !

With $x^{(r)}$ for $r = 1, ...R$ of mean x^b and variance σ_b^2:

Compute $y(r) = \mathcal{G}\left(x^{(r)}\right)$ and $y^b = \mathcal{G}\left(x^b\right)$

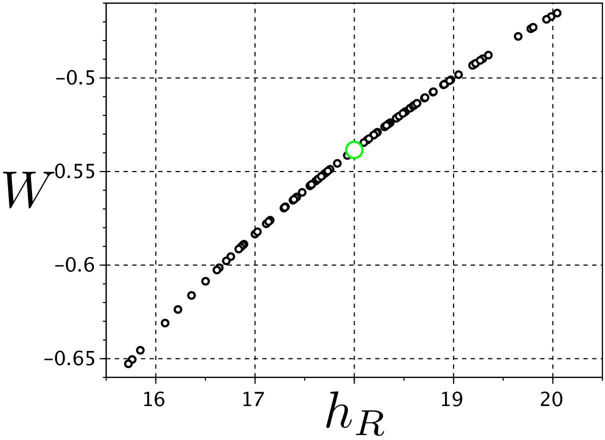
$$\sigma_b^2\, G \sim \mathcal{A}^{\sigma_b^2\, G} = \frac{1}{R} \sum_{r=1}^{R} \left(x^{(r)} - x^b\right) \left(y^{(r)} - y^b\right)$$

$$\sigma_b^2\, G^2 \sim \mathcal{A}^{\sigma_b^2\, G^2} = \frac{1}{R} \sum_{r=1}^{R} \left(y^{(r)} - y^b\right) \left(y^{(r)} - y^b\right)$$

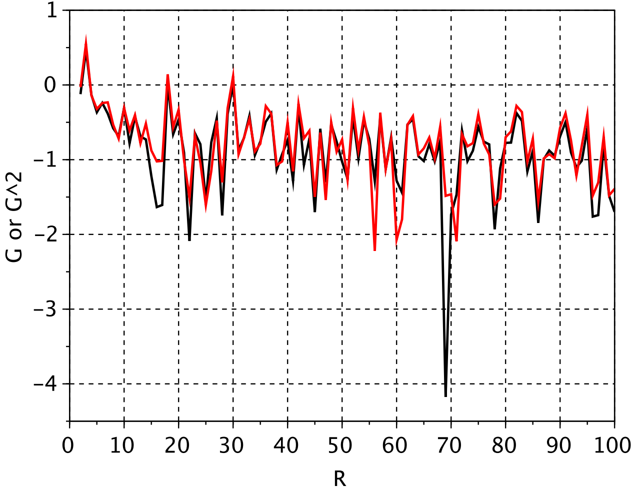
This ensemble method approximates $G = \mathcal{G}'(x^b)$:

$$\left(y^{(r)} - y^b\right) = \left[\mathcal{G}\left(x^{(r)}\right) - \mathcal{G}\left(x^b\right)\right] \sim G\, \left(x^{(r)} - x^b\right)$$

Dispersion of the ensemble used to estimate the gain

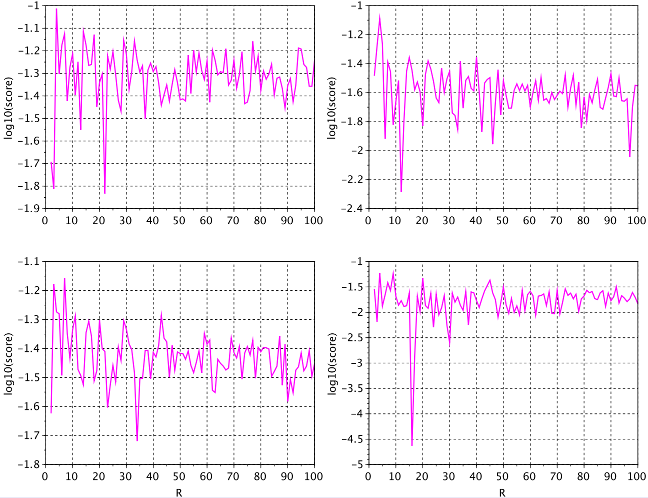


Approximation of G and G^2 by the ensemble method



Estimation of $\sigma_b^2\, G$ (black) and $\sigma_b^2\, G^2$ (red)

Validity of the ensemble method of the bore example

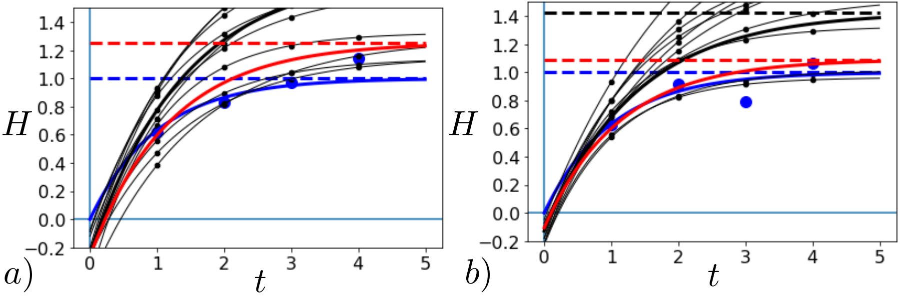


The score is $|x^a/x^t - 1|$ and R is the number of members.

Ensemble method for the assimilation of (α, P)

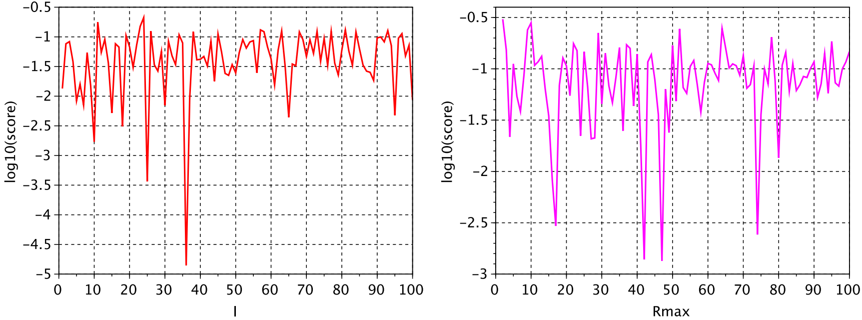
Ensemble method

Approximations $\underline{\underline{B}}\,\underline{\underline{G}}^T \sim \underline{\underline{A}}^{BG^T}$ and $\underline{\underline{G}}\,\underline{\underline{B}}\,\underline{\underline{G}}^T \sim \underline{\underline{A}}^{GBG^T}$ with random draws $\underline{x}^{(r)} = (\alpha^r, P^r)$ for $r = 1, ...R$



Red: analysis, Blue:true state, Black: background

Incremental versus ensemble method



Left (Red): Incremental for $I = 100$ draws

Right (Magenta): Ensemble for one draw as a function of R

5. ENSEMBLE METHODS IN R^N

Monte-carlo experiment

Use a great number of experiments :

- To approximate gradients

Computation of the gain matrix with an ensemble method

Compute $\underline{\underline{K}} = \underline{\underline{B}}\,\underline{\underline{G}}^T \left(\underline{\underline{G}}\,\underline{\underline{B}}\,\underline{\underline{G}}^T + \underline{\underline{R}}\right)^{-1}$ ... without computing $\underline{\underline{G}}$!

Let $\underline{x}^{(r)}$ for $r = 1, \ldots R$ of mean $\underline{x}^b$ and covariance variance $\underline{\underline{B}}$:

Compute $\underline{y}^{(r)} = \mathcal{G}\left(\underline{x}^{(r)}\right)$ and $\underline{y}^b = \mathcal{G}\left(\underline{x}^b\right)$

$$\underline{\underline{B}}\,\underline{\underline{G}}^T \sim \underline{\underline{A}}^{BG^T} = \frac{1}{R}\sum_{r=1}^{R}\left(\underline{x}^{(r)} - \underline{x}^b\right)\left(\underline{y}^{(r)} - \underline{y}^b\right)^T$$

$$\underline{\underline{G}}\,\underline{\underline{B}}\,\underline{\underline{G}}^T \sim \underline{\underline{A}}^{GBG^T} = \frac{1}{R}\sum_{r=1}^{R}\left(\underline{y}^{(r)} - \underline{y}^b\right)\left(\underline{y}^{(r)} - \underline{y}^b\right)^T$$

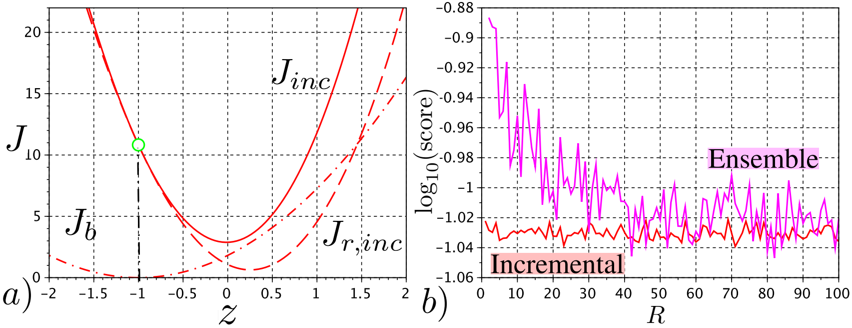
This method approximates the Jacobian matrix $\underline{\underline{G}}$ of $\mathcal{G}$ at $\underline{x}^b$:

$$\left(\underline{y}^{(r)} - \underline{y}^b\right) = \left[\mathcal{G}\left(\underline{x}^{(r)}\right) - \mathcal{G}\left(\underline{x}^b\right)\right] \sim \underline{\underline{G}}\left(\underline{x}^{(r)} - \underline{x}^b\right)$$

Exemple with $N = 30$ and $M = 10$

Observation operator $\mathcal{G} = (\mathcal{G}_1^T, ...\mathcal{G}_M^T)^T$ and other parameters

$\mathcal{G}_i(\underline{x}) = [\underline{x} - 10 * \sin(i)\, \underline{e}_i]^2$ where $(\underline{e}_1, ..., \underline{e}_N)$ is the canonical basis.
True state: $\underline{x}^t = \sum_{j=1}^N \underline{e}_j$. Covariance matrices: $\underline{\underline{B}} = 0.1\, \underline{\underline{I}}$ and $\underline{\underline{R}} = \underline{\underline{I}}$



Plot of J_{inc}, J_b and $J_{r,inc}$ as a function of z for $\underline{x} = \underline{x}^a + z\,(\underline{x}^a - \underline{x}^b)$

Homework B : Ensemble method on a simples examples

See Jupyter notebooks from: `https://olivier-thual.fr/130202`

Example in $\boldsymbol{R}$

1. Read Sections 1.1 and 1.2 of Chapter 4
2. Program both the incremental and ensemble method for one example

Example in $\boldsymbol{R}^n$

1. Read Sections 1.4 of Chapter 4
2. Program an ensemble method for a new example in $\boldsymbol{R}^n$, to be designed

Requirements

- Program and present two examples
- A written report : 5 pages minimum
- A link to Jupyter notebooks on Moodle before the dealine